Floor diagrams and refined invariants in positive genus

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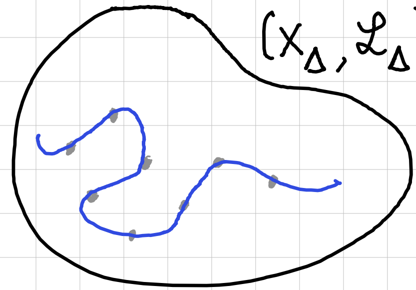
Context

* <u>enumerative toric geometry ($\mathbb{C}$)</u>

polygon Δ $\longrightarrow$ $(X_\Delta, \mathcal{L}_\Delta)$



configuration of points on X_Δ

genus 0 curves

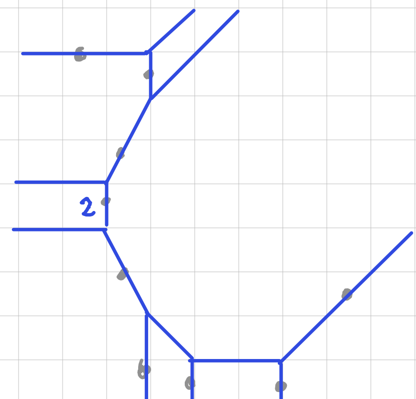
$N_0(\Delta)$

$\|$

$N_0^{trop}(\Delta)$

* <u>tropical geometry</u> : Mikhalkin's correspondence theorem

polygon Δ $\longrightarrow$



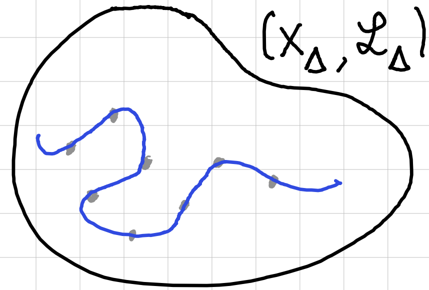
configuration of points on $\mathbb{R}^2$

genus 0 tropical curves + multiplicity

Context

* enumerative toric geometry $(\mathbb{R})$

polygon Δ $\longrightarrow$



$(X_\Delta, \mathcal{L}_\Delta)$

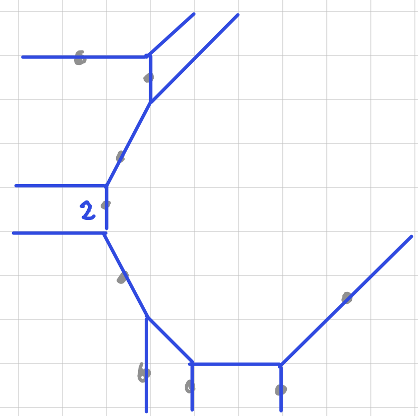
real configuration of points on X_Δ

genus 0 curves + sign

$$W(\Delta) = W^{trop}(\Delta)$$

* tropical geometry : Mikhalkin's correspondence theorem

polygon Δ $\longrightarrow$



configuration of points on $\mathbb{R}^2$

genus 0 tropical curves + real multiplicity

Refined invariants

* <u>Block-Göttsche ('16)</u> :

genus 0 tropical curves + refined multiplicity $\to G_0(\Delta) \in \mathbb{Z}[q^{\pm 1}]$.

* <u>Göttsche-Schroeter ('19)</u> :

$$\text{real configuration of points} \begin{cases} \nearrow \text{ real points} \\ \searrow \text{ pairs of complex conjugated points (number: } s) \end{cases}$$

genus 0 tropical curves + refined multiplicity $\to G_0(\Delta, s) \in \mathbb{Z}[q^{\pm 1}]$.

* $q = 1$: some relative / descendant invariants

$q = -1$: Welschinger invariant (with s pairs of $\mathbb{C}$-conjugated points)

Result

* floor diagram = combinatorial algorithm to compute $N_0(\Delta)$, $W(\Delta)$, $G_0(\Delta, s)$.
  (Brugallé-Mikhalkin '07)

* $G_0(\Delta, s)$: – choice of a pairing S of order s
  – multiplicity $\mu_S(\mathcal{D})$
  – $G_0(\Delta, s) = \sum_{\mathcal{D}} \mu_S(\mathcal{D})$ independent of S.

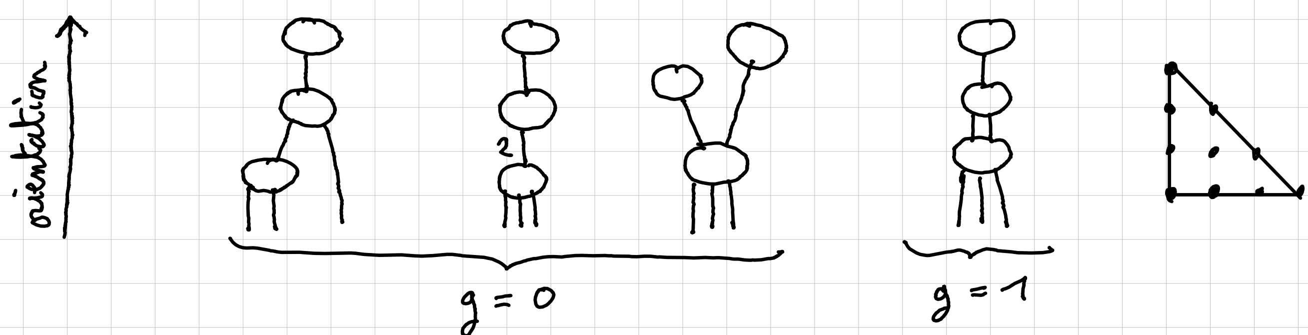
Theorem (M. '24). For any genus $g \geq 0$, one can define

$$G_g(\Delta, s) = \sum_{\mathcal{D}} \mu_S(\mathcal{D}).$$

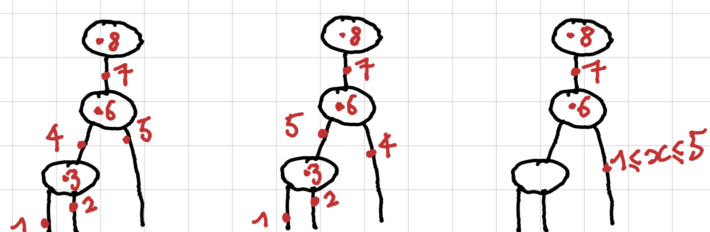
Simultaneously : Shustin-Sinichkin '24.

Floor diagrams

* floor diagram with Newton polygon Δ : connected, oriented acyclic, weighted graph ; # of vertices and edges prescribed by Δ ; other condition.



* marking : increasing bijection $m : \mathcal{D} \to \{1, \dots, \#\mathcal{D}\}$.

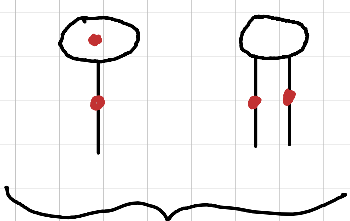


Floor diagrams

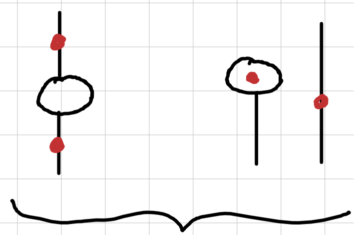
* **pairing S** : s disjoint pairs of successive integers of $\{1, \dots, \#\mathcal{D}\}$.

S is **compatible** if : $\forall$ pair $\alpha \in S$, $m^{-1}(\alpha)$ is :

- $\{e, v\}$ with e and v adjacent,
- $\{e, e'\}$ with $\exists v$, e and e' both enter or leave v.



compatible



not compatible

Floor diagrams

* S- multiplicity : $E_0 = \{ e \mid e \text{ unpaired}\}$

$$E_1 = \{ e \mid \exists v,\ e \text{ and } v \text{ are paired}\}$$

$$E_2 = \{ \{e,e'\} \mid e \text{ and } e' \text{ are paired}\}$$

$n \in \mathbb{Z}$: $$[n](q) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \in \mathbb{Z}[q^{\pm 1/2}]$$

$$\boxed{\mu_S(\mathcal{D},m) = \prod_{e\in E_0} [w(e)]^2 \prod_{e\in E_1} [w(e)](q^2) \prod_{\{e,e'\}\in E_2} \frac{[w(e)][w(e')][w(e)+w(e')]}{[2]}.}$$

* $$G_0(\Delta,s) = \sum_{(\mathcal{D},m)} \mu_S(\mathcal{D},m).$$

Elements of proof

Theorem. $\forall$ genus $g \geqslant 0$, $\sum_{(\mathcal{D},m)} \mu_S(\mathcal{D},m)$ does not depend on the pairing S as long as it has order s : $G_g(\Delta, s)$.

$*$ S and S' such that $S \setminus \{i, i+1\} = S' \setminus \{i+1, i+2\}$.

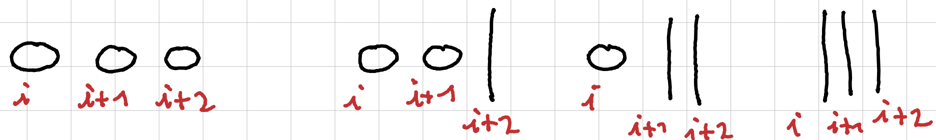
$*$ For any $(\mathcal{D},m)$, find a subset $P \subset \{\text{marked diagrams}\}$ such that :

$(\mathcal{D},m) \in P$ and $\sum_P \mu_S = \sum_P \mu_{S'}$.

$*$ Fix $(\mathcal{D},m)$, look at the elements marked i, $i+1$, $i+2$.

Elements of proof

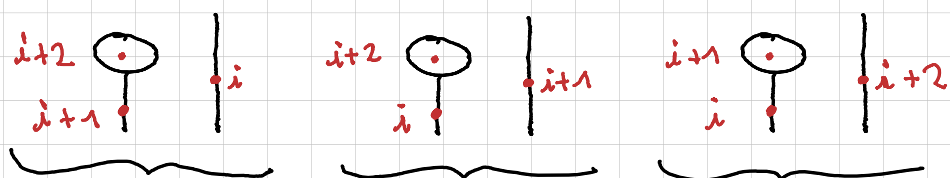
* Easy cases :



S and S' incompatible.

$\mu_S(\mathcal{D},m)=\mu_{S'}(\mathcal{D},m)=0$, $\quad P=\{(\mathcal{D},m)\}$.

* More interesting :



$(\mathcal{D},m_0)$ $\quad(\mathcal{D},m_1)$ $\quad(\mathcal{D},m_2)$

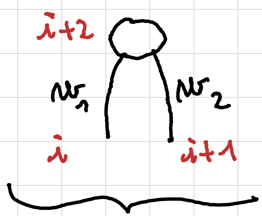
then $\mu_S(\mathcal{D},m_0)=\mu_S(\mathcal{D},m_1)=\mu_{S'}(\mathcal{D},m_1)=\mu_{S'}(\mathcal{D},m_2)=0$

and $\mu_S(\mathcal{D},m_2)=\mu_{S'}(\mathcal{D},m_0)$.

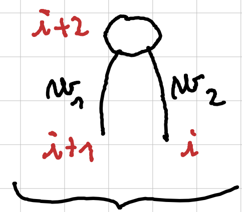
$\rightarrow P=\{(\mathcal{D},m_k),\ 0\leq k\leq 2\}$.

Elements of proof

* **Another one :**



$(\mathcal{D}, m_0)$



$(\mathcal{D}, m_1)$

then $$\mu_S(\mathcal{D}, m_0) = \mu_S(\mathcal{D}, m_1) = \frac{[\varpi_1][\varpi_2][\varpi_1+\varpi_2]}{[2]} \times (\dots)$$

and $\mu_{S'}(\mathcal{D}, m_0) = [\varpi_1]^2 [\varpi_2](q^2) \times (\dots)$, $\mu_{S'}(\mathcal{D}, m_1) = [\varpi_1](q^2)[\varpi_2]^2 \times (\dots)$

Elementary calculation $\rightarrow$ $P = \{(\mathcal{D}, m_0), (\mathcal{D}, m_1)\}$.

Similar works

* Schroeter - Shustin '18 : $g=1$, tropical curves

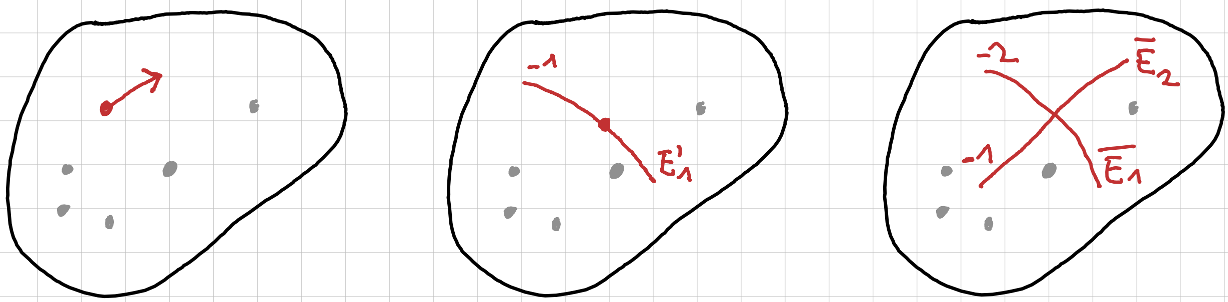
* Shustin - Sinichkin '24 : $g \geq 1$, tropical curves

prop. $G_g(\Delta, s)$ correspond to the invariants of [SS24] .

* Shustin - Sinichkin '24 : $g=1$ $\rightsquigarrow$ # curves through some points and tangent to some lines at some points.

s tangency conditions

On characteristic numbers



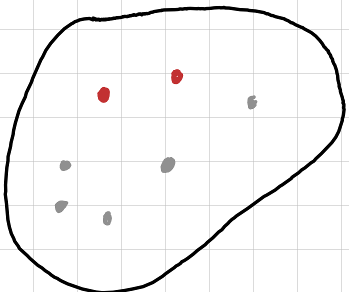
$X = X_\Delta$ $\qquad$ X' $\qquad$ $\bar{X}$

$$\#\left\{\begin{array}{l}\text{curves on } X \text{ of genus } g,\\ \text{through the } \bullet \text{ and with } \nearrow\end{array}\right\} = \#\left\{\begin{array}{l}\text{curves on } \bar{X} \text{ of genus } g,\\ \text{through the } \bullet\text{, intersecting } \bar{E}_2\\ \text{but not } \bar{E}_1\end{array}\right\}$$

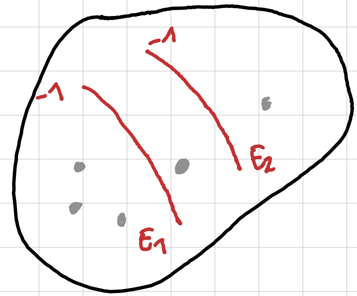
$$\| $$

$$N_g(\bar{X}, \Delta - \bar{E}_1 - 2\bar{E}_2)$$

On characteristic numbers



$X = X_\Delta$



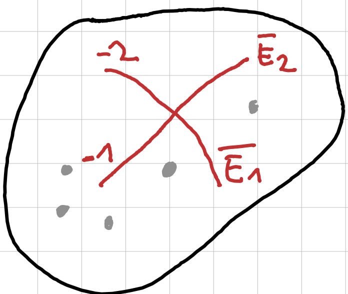
$\tilde{X}$

$$\#\left\{\begin{array}{l}\text{curves on } X \text{ of genus } g,\\ \text{through the } \bullet \text{ and } \bullet\end{array}\right\} = \#\left\{\begin{array}{l}\text{curves on } \tilde{X} \text{ of genus } g,\\ \text{through the } \bullet,\ \text{intersecting } E_1\\ \quad \text{and } E_2\end{array}\right\}$$

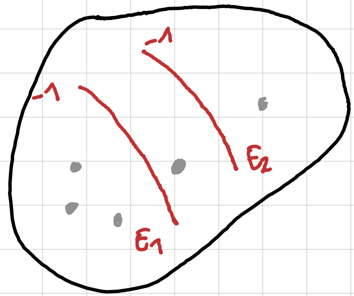
$$\| $$

$$N_g(\tilde{X}, \Delta - E_1 - E_2)$$

On characteristic numbers



$\overline{X}$



$\tilde{X}$

* Abramovich-Bertram formula :

$$\underbrace{N_g(\overline{X}, \Delta - \overline{E}_1 - 2\overline{E}_2)}_{\text{tangency condition } (s=1)} = \underbrace{N_g(\tilde{X}, \Delta - E_1 - E_2) - 2N_g(\tilde{X}, \Delta - 2E_1)}_{\text{no tangency condition } (s=0)}$$

$\hookrightarrow \; G_g(\Delta, 0) \; (q=1)$

$$\boxed{\text{prop. } \; G_g(\Delta, s+1) = G_g(\Delta, s) - 2G_g(\tilde{\Delta}, s) \, .}$$